Band of localized electromagnetic waves in random arrays of dielectric cylinders

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Anderson localization of electromagnetic waves in random arrays of dielectric cylinders is studied. An effective theoretical approach based on analysis of probability distributions, not averages, is developed. The disordered dielectric medium is modeled by a system of randomly distributed two-dimensional electric dipoles. Spectra of certain random matrices are investigated and the appearance of the band of localized waves emerging in the limit of an infinite medium is discovered. It suggests deeper insight into the existing experimental results. [S1063-651X(97)00410-8]

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Disordered dielectric structures with typical length scale matching the wavelength of electromagnetic radiation are still a subject of intensive experimental and theoretical studies. Electromagnetic waves propagating in these structures mimic, to reasonable extent, the behavior of electrons in disordered semiconductors. Many ideas concerning transport properties of light and microwaves in such media exploit the well-developed theoretical methods and concepts of solid-state physics. Let us mention, e.g., the concept of electron localization in noncrystalline systems such as amorphous semiconductors or disordered insulators. According to Anderson [1], an entire *band* of electronic states can be spatially localized in a sufficiently disordered infinite material. Before this discovery, it was believed that electronic states in infinite media are either extended, by analogy with the Bloch picture for crystalline solids, or are localized around *isolated* spatial regions such as surfaces and impurities [2].

Usually experiments related to electron localization deal with such measurable quantities as transmission, diffusion constant, or transport coefficient. The natural quantity to look for is, e.g., the static (dc) conductivity. Intuitively, localized states are basically bound to stay in a finite region of space for all times, whereas extended ones are free to flow out of any finite region. Therefore, it is natural to expect that the material in which an entire band of electronic states is localized will be an insulator, whereas the case of extended states will correspond to a conductor. In this way the phenomenon of Anderson localization may be related to a dramatic inhibition of the propagation of an electron when it is subject to a spatially random potential. Although this connection is not proven on a general basis, it is certainly valid within reasonable physical models [3].

A very common theoretical approach in investigations of Anderson localization in solid-state physics is to study the transport equation for the ensemble-averaged squared modulus of the wave function [4–6]. Under some assumptions such a transport equation can be converted into a diffusion equation. The diffusion constant D becomes a parameter monitoring behavior of the system. Strong localization is achieved when the diffusion constant in the scattering medium becomes zero. When the fluctuations of the electronic static potential become large enough, the wave function ceases to diffuse and becomes localized. Thus the Anderson transition may be viewed as a transition from particlelike behavior described by the diffusion equation to wavelike behavior described by the Schrödinger equation which results in localization [7].

It is commonly believed that the Anderson localization is completely based on the interference effects in multiple elastic scattering. It is obvious, however, that interference is a common property of all wave phenomena. Indeed, many generalizations of electron localization to electromagnetic waves have been proposed [7–13]. Weak localization of electromagnetic waves, meaning enhanced coherent backscattering, is presently relatively well understood theoretically [14–16] and confirmed by experiments [17–19]. A different interesting problem is whether interference effects in disordered dielectric media can reduce the diffusion constant down to zero leading to strong localization. Despite the observation of scale dependence of the diffusion constant $D(L)$ in such media (which may be considered as a reasonable indication of the Anderson transition) there still is no convincing experimental demonstration that strong localization could be possible in three-dimensional disordered dielectric structures. Such a demonstration has only been given for two dimensions, where strong localization takes place for arbitrarily small value of the mean free path (if the medium is sufficiently large). The strongly scattering medium has been provided by a set of dielectric cylinders randomly placed between two parallel aluminum plates on half the sites of a square lattice [20].

There still is a need for sound theoretical models providing deeper insight into this interesting effect. Such models should be based directly on the Maxwell equations and they should be simple enough to provide calculations without too many approximations. There is a temptation to immediately apply averaging procedures as soon as ‘‘disorder’’ is introduced into the model. Averaging of the scattered intensity over some random variable leads to a transport theory [21]. But ‘‘there is a very important and fundamental truth about random systems we must always keep in mind: no real atom is an average atom, nor is an experiment done on an ensemble of samples’’ [22]. We always deal with a specific example of the disordered system. Therefore what we really need to properly understand the existing experimental results are probability distributions, not averages. In this paper we develop a theoretical model of the Anderson localization of electromagnetic waves in two-dimensional (2D) dielectric media without using any averaging procedures.

In the following we study the properties of the stationary solutions of the Maxwell equations in two-dimensional media. This means that one (z) out of three dimensions is translationally invariant and only the remaining two ($\vec{\rho}$) are random. The main advantage of two-dimensional localization is that we can use the scalar theory of electromagnetic waves $\vec{E}(\vec{r},t)=\mathrm{Re}\{\vec{e}_z\mathcal{E}(\vec{\rho})e^{-i\omega t}\}$ and still can try to compare, at least qualitatively, the model predictions with experimental results [23]. Consequently, the polarization of the medium takes the form: $\vec{P}(\vec{r},t)=\mathrm{Re}\{\vec{e}_z\mathcal{P}(\vec{\rho})e^{-i\omega t}\}$.

Localization of electromagnetic waves in 2D media is studied experimentally in microstructures consisting of dielectric cylinders with diameters and mutual distances being comparable to the wavelength [20]. It is a reasonable assumption that what really counts for the basic features of localization is the scattering cross section and not the real geometrical size of the scatterer. Therefore we will represent the dielectric cylinders located at the points $\vec{\rho}_a$ by *single* 2D electric dipoles $\mathcal{P}(\vec{\rho})=\Sigma_a p_a \delta^{(2)}(\vec{\rho}-\vec{\rho}_a)$. Although this approximation is strictly justified only when the diameter of the cylinders is much smaller than the wavelength, in practical calculations many multiple-scattering effects can be obtained qualitatively for coupled electrical dipoles [24–26].

The point-scatterer approximation requires a representation for the scatterers that fulfills the optical theorem rigorously and conserves energy in the scattering processes. To satisfy the conservation of energy, the dipole moments p_a should be coupled to the electric field of the incident wave $\mathcal{E}'(\vec{\rho}_a)$ by complex "polarizability" $(e^{i\phi}-1)/2$, which can take values from a circle on the complex plane [23]

$$i\pi k^2 p_a=\frac{1}{2}(e^{i\phi}-1)\mathcal{E}'(\vec{\rho}_a), \tag{1}$$

where $k=\omega/c$ is the wave number in vacuum. The field acting on the ath cylinder is the sum of some free field $\mathcal{E}^{(0)}(\vec{\rho})$, which obeys the Maxwell equations in vacuum, and waves scattered by all *other* cylinders [23]

$$\mathcal{E}'(\vec{\rho}_a)=\mathcal{E}^{(0)}(\vec{\rho}_a)+2k^2\sum_{b\neq a}K_0(-ik|\vec{\rho}_a-\vec{\rho}_b|)p_b, \tag{2}$$

where K_0 denotes the modified Bessel function of the second kind.

To get some insight into the physical meaning of the parameter ϕ from Eq. (1) let us observe that it is directly related to the total scattering cross section σ of an individual dielectric cylinder represented by the single dipole

$$k\sigma=2(1-\cos\phi). \tag{3}$$

Therefore ϕ is a function of the frequency ω and physical parameters describing the cylinders such as dielectric constant ϵ and radius R. Thus each choice of ϕ is in fact a choice of scatterers.

Inserting Eq. (1) into Eq. (2) we obtain the system of linear equations

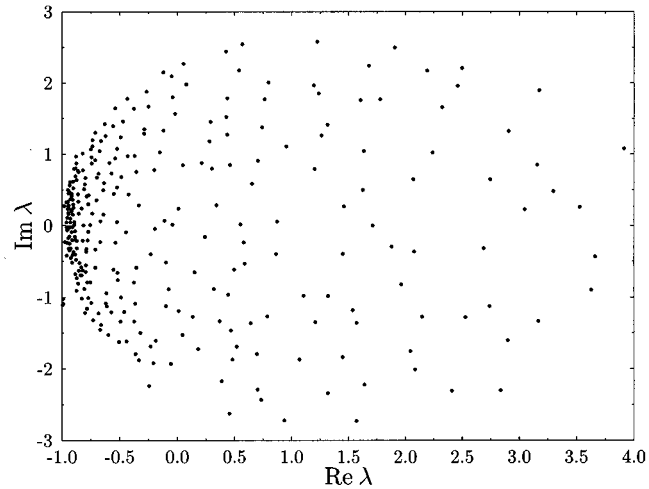

FIG. 1. Spectrum λ_j of G matrix (diagonalized numerically) corresponding to a certain specific configuration of $N=300$ cylinders placed randomly inside a circle, with uniform density $n=1$ cylinder per wavelength squared. Most eigenvalues are located near the $\mathrm{Re}\lambda=-1$ axis, fulfilling the localization condition (7) almost exactly.

$$\sum_b M_{ab}\mathcal{E}'(\vec{\rho}_b)=\mathcal{E}^{(0)}(\vec{\rho}_a), \tag{4}$$

determining the field acting on each cylinder $\mathcal{E}'(\vec{\rho}_a)$ for a given field of the free wave $\mathcal{E}^{(0)}(\vec{\rho}_a)$ incident on the system. If we solve it and use again Eq. (1) to find p_a, then we are able to find the electromagnetic field everywhere in space $\mathcal{E}(\vec{\rho})=\mathcal{E}^{(0)}(\vec{\rho})+2k^2\Sigma_a K_0(-ik|\vec{\rho}-\vec{\rho}_a|)p_a$ (for $\vec{\rho}\neq\vec{\rho}_a$). A similar integral equation relating the stationary outgoing wave to the stationary incoming wave is known in general scattering theory as the Lippmann-Schwinger equation [27].

Now, let us study eigenvalues λ_j of the matrix

$$i\pi G_{ab}=\begin{cases}2K_0(-ik|\vec{\rho}_a-\vec{\rho}_b|) & \text{for } a\neq b,\\ 0 \quad \text{for } a=b,\end{cases} \tag{5}$$

which depend only on the positions of the cylinders $k\vec{\rho}_a$ scaled in wavelengths. To gain some information about the spectrum of the G matrix (5) corresponding to systems of *infinite* number of cylinders placed randomly with uniform density, we have to study the properties of *finite* systems for increasing number of dipoles N (while keeping the density constant). First, in Fig. 1 we plot the spectrum λ_j of a G matrix (diagonalized numerically) corresponding to a certain specific configuration of $N=300$ cylinders placed randomly inside a circle, with the uniform density $n=1$ cylinder per wavelength squared. We see that quite a lot of eigenvalues are located near the $\mathrm{Re}\lambda=-1$ axis. As will be discussed below, this is a universal property of 2D G matrices, not restricted to this specific realization of the system only. Moreover, the number of eigenvalues fulfilling (to reasonable accuracy) the above condition increases with N. To prove these statements we diagonalize numerically the G matrix (5) for 10^3 different distributions $\vec{\rho}_a$ of *finite* number of N cylinders. Then we construct a two-dimensional

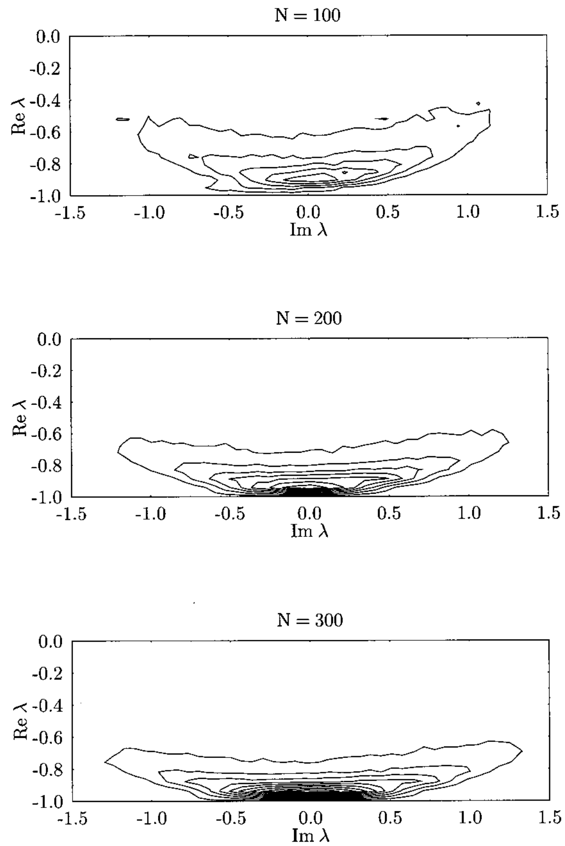

FIG. 2. Contour plots of the density of eigenvalues $P(\lambda)$ calculated for 10^3 different distributions of cylinders for increasing size of the system $N=100,200,300$. The distance between neighboring contours is the same for all plots. For increasing values of N, the probability distribution $P(\lambda)$ apparently moves towards the $\mathrm{Re}\lambda=-1$ axis and, simultaneously, its variance along the $\mathrm{Im}\lambda=\mathrm{const}$ axes decreases.

histogram of eigenvalues λ_j from all distributions. It approximates the corresponding probability distribution $P(\lambda)$ which is normalized in the standard way $\int d^2\lambda P(\lambda)=1$. Let us now compare the contour plots of $P(\lambda)$ for different numbers of cylinders $N=100,200,300$. They are presented in Fig. 2. For convenience, the distance between neighboring contours is the same for all plots. It follows from inspection of Fig. 2 that for increasing values of N, the probability distribution $P(\lambda)$ apparently moves towards the $\mathrm{Re}\lambda=-1$ axis and, simultaneously, its variance along the $\mathrm{Im}\lambda=\mathrm{const}$ axes decreases. In addition, Fig. 3 we have the surface plot of the function $P(\lambda)$ calculated for the case $N=300$. It clearly shows that for all configurations (without, maybe, a set of zero measure) most eigenvalues are located near the $\mathrm{Re}\lambda=-1$ axis. This tendency is more and more pronounced with increasing size of the system measured by N. Our numerical investigations indicate that in the limit of an *infinite* medium, the probability distribution under consideration will tend to the δ function

$$\lim_{N\to\infty} P(\lambda)=\delta(\mathrm{Re}\lambda+1)f(\mathrm{Im}\lambda). \tag{6}$$

This means that in this limit for almost any random distribution of the cylinders $k\vec{\rho}_a$, infinite number of eigenvalues satisfies the condition

$$\mathrm{Re}\lambda_j=-1. \tag{7}$$

We have some numerical evidence that this fact is a general property of G matrices, not restricted to the considered case of one dipole per wavelength squared $n=1$ only [although the function f from Eq. (6) certainly may depend on n].

It is important to note that a deep connection exists between electromagnetic waves localized in the system of dielectric cylinders located at points $k\vec{\rho}_a$ and eigenvectors of the G matrix which correspond to eigenvalues λ_j satisfying the localization condition (7). To see this connection it is enough to observe that eigenvectors of the G matrix are simultaneously eigenvectors of the M matrix from Eq. (4). The difference is that contrary to the G matrix, the M matrix depends not only on the positions of the cylinders $k\vec{\rho}_a$ but also on the parameter ϕ. If a certain eigenvalue λ_j of the G matrix obeys the condition (7) then we may choose the parameter ϕ in such a way that the corresponding eigenvalue of the M matrix will be equal to zero, i.e., $\Lambda_j(\phi_j)=(1+\lambda_j/2)-e^{i\phi_j}(\lambda_j/2)=0$. It is enough to take

$$\phi_j=\arg\left(1+\frac{\lambda_j}{2}\right)-\arg\left(\frac{\lambda_j}{2}\right). \tag{8}$$

Thus the corresponding eigenvector is a nonzero solution of Eqs. (4) for the incoming wave equal to zero. Therefore it may be interpreted as a localized wave [23]. The physical meaning of this choice of ϕ is clear. For given radiuses R and dielectric constants $\epsilon(\omega)$ of the cylinders the parameter ϕ from Eq. (1) is fixed but remains a function of the frequency, i.e., $\phi=\phi(\omega)$. Thus for a given frequency and given positions of the scatterers we can always find such dielectric cylinders, that a localized wave exists in the medium.

Now consider a certain realization of an infinite system of cylinders located randomly at points $\vec{\rho}_a$ with uniform density η. As pointed out before infinite number of eigenvalues λ_j of

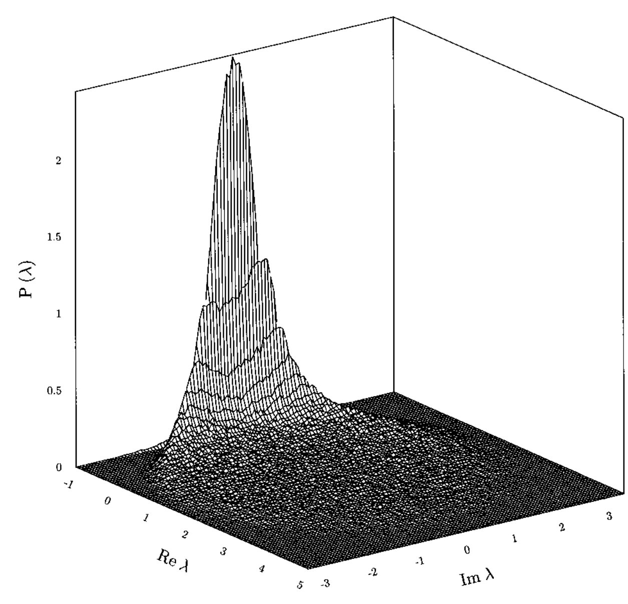

FIG. 3. Surface plot of $P(\lambda)$ corresponding to the case $N=300$ from Fig. 2. It shows where the most weight of the $P(\lambda)$ distribution is located.

the G matrix satisfies the condition (7). This occurs not only for $n=1$ but for a whole range of n and therefore, for fixed physical density $\eta = k^2 n/(2\pi)^2$, for a range of frequencies ω. Thus the values of ϕ_j given by Eq. (8) which correspond to localized waves should be now regarded as functions of ω. This means, that in the system considered localized waves appear at *discrete* frequencies ω_j determined by the crossing points of the function $\phi(\omega)$ with infinite number of curves $\phi_j(\omega)$.

It is reasonable to expect that in the case of a *random* and *infinite* system a countable set of frequencies ω_j corresponding to localized waves becomes dense in some finite interval. Therefore, an entire *band* of spatially localized electromagnetic waves appears. Anderson localization occurs when this happens. Physically speaking this means that different realizations of sufficiently large system of randomly placed cylinders are practically (i.e., by a transmission experiment) indistinguishable from each other. For example, for a certain realization of a random and infinite one-dimensional system one can prove mathematically [28] that incident waves are totally reflected for ''almost any'' energy, i.e., except the discrete set (of zero measure) for which the transmission is equal to unity. This dense set of discrete energies exceptional in the Furstenberg theorem [29] corresponds to the band of localized waves.

According to the scaling theory of localization [30], the dimension of the disordered medium is a crucial parameter. In one and two dimensions any degree of disorder will lead to localization, while in three dimensions a certain critical degree of disorder is needed before localization will set in. Our calculations do not exclude the possibility that in an *infinite* 2D medium the band of localized waves may appear for $\phi \rightarrow 0$ (or $\mathrm{Im}\lambda \rightarrow \pm\infty$). However, in all experiments we can investigate only systems confined to certain *finite* regions of space. As follows from Fig. 2 (dealing with finite media), with increasing size of the system the band of localized waves appears *faster* for $|\phi| \simeq \pi$ (or $\mathrm{Im}\lambda \simeq 0$), than for other values of ϕ. This means that the scattering cross section of individual scatterers (3) should be made maximal (for example, by tuning the frequency to match the internal resonances of the cylinders). This is not necessarily the case for three-dimensional media.

In summary, we have developed an effective theoretical approach (based on analysis of probability distributions) to Anderson localization of electromagnetic waves in random arrays of dielectric cylinders. Investigating spectra of two-dimensional random Green matrices we have actually observed numerically the appearance of the *band* of localized electromagnetic waves emerging in the limit of the infinite medium. Similar approach can provide deeper insight into the existing experimental results concerning localization in 3D media. This interesting problem will be addressed in detail in a forthcoming paper.

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